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On the Motzkin–Grünbaum theorem

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A constructive proof of the Motzkin–Grünbaum theorem (at least one fullerene C_n with any even $n \geq 24$ exists) is suggested.

1. Introduction

We consider a fullerene C_n as any convex polyhedron built from penta- and hexagonal facets touching at three of any of n vertices. The simplest fullerene C_{20} is nothing but the dodecahedron. But an attempt to build a fullerene C_{22} with one hexagon immediately leads to the C_{24} form with two hexagons (Fig. 1). The Motzkin–Grünbaum theorem states that at least one fullerene with any even $n \geq 24$ exists (Grünbaum & Motzkin, 1963). Our proof of the theorem is a constructive one. We start from the 'lemma on the contour' (Voytekhovsky & Stepenshchikov, 2004).

2. Proof

To prove the theorem, we follow the idea used in Klein & Liu (1992; Voytekhovsky & Stepenshchikov, 2004) and provide examples of the fullerenes. In fact, we suggest some initial fullerenes and the algorithm to build their endless series, which cover, in total, any even $n \geq 24$.

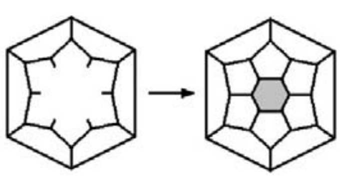

Figure 1
The impossible attempt to build the Schlegel projection of a C_{22} fullerene.

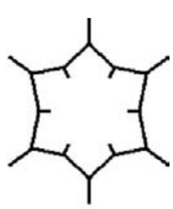

Figure 2
The 'gear' contour.

The following lemma was proved in Voytekhovsky & Stepenshchikov (2004). *Let us consider any closed contour built from the edges of a fullerene with e_{in} and e_{out} being the numbers of edges touching it from inside and outside, respectively. Then, the number f_5 of pentagons inside a contour equals $6 + e_{in} - e_{out}$ regardless of the number f_6 of hexagons*. Fig. 2 shows the 'gear' contour with $e_{\rm in} = e_{\rm out} = 6$. Hence, $f_5 = 6$ for it.

Now, we construct the initial fullerenes, each one from two caps bounded by the 'gear'. When constructing the caps, we put at least one pentagon at the contour. Otherwise we could delete a belt of six hexagons and obtain the 'gear' contour. All the ways to fill up the 'gear' with six pentagons (white), any number of hexagons (black) and the above restriction are given in Fig. 3.

Composing two caps from Fig. 3, one may construct the fullerenes with 14 to 24 facets and 24 to 44 vertices, respectively. (For any simple – three edges meet at each vertex – polyhedron, the numbers f of facets and v of vertices relate to each other as follows: $v = 2f - 4$.) In order to prove the theorem, we need the fullerenes C_{24}–C_{34} only. All of them are given in Fig. 4.

Finally, to build six endless series, we insert, one by one, the belts of six hexagons between the caps of the initial fullerenes (Fig. 5). As 6 additional facets lead to 12 additional vertices of the fullerene, the initial series C_{24}–C_{34} leads to C_{36}–C_{46}, C_{48}–C_{58}, C_{60}–C_{70} *etc*. Thus, at least one fullerene C_n with any even $n \geq 24$ exists.

3. Conclusions

The most important theorems on the fullerenes are proved in a constructive way. Simultaneously, the desired structures are visualized. This is a positive moment for various physical applications. But the non-existence of the fullerene C_{22} is proved in the same way. It is an intriguing task to prove its non-existence in an algebraical way.

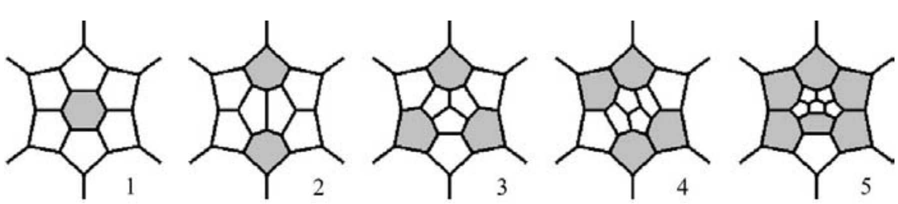

Figure 3
All the ways to fill up the 'gear' with at least one pentagon at the contour.

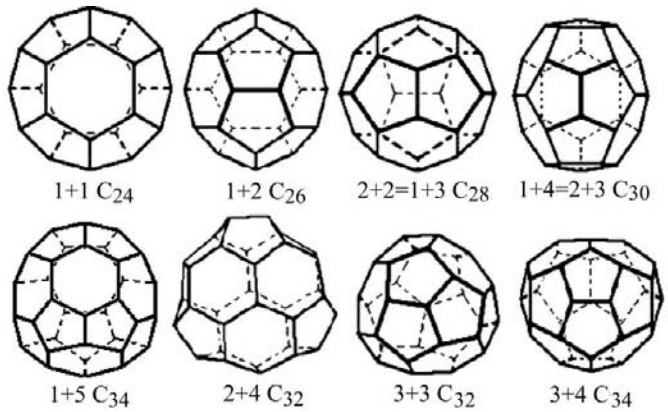

Figure 4
The initial fullerenes C_{24}–C_{34}. The numbers relate to Fig. 3: 1+1 ($\overline{12}m2$), 1+2 ($\bar{6}m2$), 1+3 ($\bar{4}3m$), 1+4 ($mm2$), 1+5 (m), 2+4 ($\bar{6}m2$), 3+3 (32), 3+4 (m).

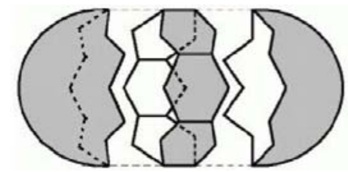

Figure 5
The belt of six hexagons between the caps of an initial fullerene (scheme).

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